# A quasi-parafermionic realization of $G_2$ and $U_q(G_2)$



1992 J. Phys. A: Math. Gen. 25 L383

(http://iopscience.iop.org/0305-4470/25/8/001)

J. Phys. A: Math. Gen. **25** (1992) L383–L388. Printed in the UK

**LETTER TO THE EDITOR**

# A quasi-parafermionic realization of $G_2$ and $U_q(G_2)$

L Frappat†‖ and A Sciarrino‡¶

† Laboratoire de Physique Théorique ENSLAPP§, Groupe d'Annecy, Chemin de Bellevue, BP 110, F-74941 Annecy-le-Vieux Cedex, France
‡ Università di Napoli 'Federico II', Dipartimento di Scienze Fisiche, and INFN, Sezione di Napoli, I-80125 Napoli, Italy

Received 12 September 1991, in final form 30 January 1992

**Abstract.** We present a construction of the exceptional Lie algebra $G_2$ and of the corresponding quantum algebra $U_q(G_2)$ using quasi-parafermionic creation and annihilation operators and their quantum analogue. As a by-product, a new realization of $U_q(A_2)$ is found.

Quantum groups, introduced in the study of integrable models, have become mathematical structures of relevant interest in many fields of physics. Explicit constructions of the quantum universal enveloping algebra, the so-called $q$-algebra, of SU(2) have been obtained by introducing the $q$-analogue of the harmonic oscillator boson operator [1]. Introducing also the $q$-analogue of the fermionic operators satisfying a $q$-deformed Clifford algebra, this construction has been generalized to the universal enveloping algebras of all classical Lie algebras [2], to the exceptional Lie algebras [3] except $U_q(G_2)$. For this last one, only an explicit matrix realization is known [4].

In this letter we want to present a construction of $G_2$ and of $U_q(G_2)$ in terms of objects which are similar to parafermions (respectively $q$-parafermions) but as they do not satisfy all the equations defining parafermions, we prefer to call them 'skedofermions' (from greek σκεδος close to) in order to avoid confusion with the already existing literature on parafermions [5] and $q$-parafermions [6].

Let us first discuss why the construction proposed in [2, 3] cannot be extended to $U_q(G_2)$. For the Lie algebra $G_2$ an explicit construction in terms of fermions can be obtained [7] by exploiting the embedding $G_2 \subset B_3 \subset D_4$. As $G_2$ is a singular subalgebra of $B_3$, a simple root of $G_2$ is a linear combination of simple roots of $B_3$, and therefore the corresponding generator is a linear combination of the generators of $B_3$. As the lack of linearity is the peculiar feature of the deformed algebras, this realization cannot be deformed in a consistent way. Let us remark that the same structure appears in the case of the Lie algebra $F_4$ which is a singular subalgebra of $E_6$. The construction of this algebra in terms of fermions exploiting the embedding $F_4 \subset E_6$ [7] cannot be deformed. So the construction of [3] can be seen as the deformation of the construction of $F_4$ obtained by the decomposition of this algebra in its maximal subalgebra $B_4$:

$$\mathbf{52} = \mathbf{36} + \mathbf{16}. \tag{1}$$

§ URA 14-36 du CNRS, associée à l'École Normale Supérieure de Lyon et au Laboratoire d'Annecy-le-Vieux de Physique des Particules (IN2P3).
‖ E-mail: Bitnet FRAPPAT@FRLAPP51.
¶ E-mail: SCIARRINO@NA.INFN.IT.

0305-4470/92/080383+06$04.50 

In the following, we shall present at first a realization of $G_2$ in terms of fermions exploiting the decomposition in its maximal subalgebra $A_2$:

$$\mathbf{14}=\mathbf{8}+\mathbf{3}+\bar{\mathbf{3}} \tag{2}$$

which is suitable to be deformed.

Let us briefly recall the realization of the algebra $A_2$ in terms of fermionic operators. Let $a_i$, $a_i^+$ ($i=1,2,3$) be fermions satisfying

$$\{a_i^+, a_j^+\}=\{a_i, a_j\}=0 \qquad \text{and} \qquad \{a_i^+, a_j\}=\delta_{ij}. \tag{3}$$

The algebra $A_2$ is spanned by (with $i,j=1,2,3$, $i\neq j$ and $k=1,2$)

$$a_i^+a_j \qquad \text{and} \qquad a_k^+a_k-a_{k+1}^+a_{k+1}=H_k. \tag{4}$$

The generators corresponding to the simple positive roots are $a_1^+a_2$ and $a_2^+a_3$ and the $H_k$ form the basis of the Cartan subalgebra. With respect to this realization, the $a_i^+$ and $a_i$ transform as the $\mathbf{3}$ and the $\bar{\mathbf{3}}$ respectively. We can get another realization of these fundamental representations in terms of bilinears in $a_i$ or $a_i^+$ using the property

$$\bar{\mathbf{3}}=(\mathbf{3}\times\mathbf{3})_{\mathrm{A}} \qquad \text{and} \qquad \mathbf{3}=(\bar{\mathbf{3}}\times\bar{\mathbf{3}})_{\mathrm{A}} \tag{5}$$

where the subscript A denotes the antisymmetric part of the Kronecker product.

We can now realize the $G_2$ algebra using as generators corresponding to the simple positive roots (index 1 is for the long root, index 2 for the short root)

$$E_1=a_1^+a_2 \qquad \text{and} \qquad E_2=a_2^+-a_1a_3 \tag{6a}$$

and generators corresponding to the simple negative roots

$$F_1=a_2^+a_1 \qquad \text{and} \qquad F_2=a_2+a_1^+a_3^+ \tag{6b}$$

with corresponding Cartan generators

$$\begin{aligned} H_1&=[E_1, F_1]=a_1^+a_1-a_2^+a_2\\ H_2&=[E_2, F_2]=2a_2^+a_2-a_1^+a_1-a_3^+a_3. \end{aligned} \tag{7}$$

These generators satisfy the usual relations in the Chevalley basis

$$[H_i, E_j]=a_{ij}E_j \qquad [H_i, F_j]=-a_{ij}F_j \tag{8}$$

where $a_{ij}$ is the $G_2$ Cartan matrix

$$\begin{pmatrix} 2 & -1\\ -3 & 2\end{pmatrix}. \tag{9}$$

It is a simple calculation to show that the Serre–Chevalley relations are verified. Indeed, they are trivially verified as each term is identically vanishing†.

The fourteen elements of $G_2$ will be (where $i,j,k\in\{1,2,3\}$ and summation over repeated indices is understood)

$$\begin{matrix} a_i^+a_j\ (i\neq j) & a_i^++\tfrac{1}{2}\varepsilon_{ijk}a_ja_k & a_i-\tfrac{1}{2}\varepsilon_{ijk}a_j^+a_k^+\\ a_1^+a_1-a_2^+a_2 & 2a_2^+a_2-a_1^+a_1-a_3^+a_3. & \end{matrix} \tag{10}$$

Let us remark that the bosonic realization of $A_2$ is not suitable to build the realization of $G_2$ as the terms $\varepsilon_{ijk}a_ja_k$ and $\varepsilon_{ijk}a_j^+a_k^+$ are zero for bosons.

In the realization of orthogonal Lie algebras in terms of creation and annihilation operators, the fundamental (spinorial) representations of the algebra is spanned by

† This happens for all the Serre–Chevalley relations for $a_{ij}<0$ when the Lie algebra is realized in terms of fermionic operators.

the Fock space, generated by the creation operators on the vacuum state $|0\rangle$ defined as the state annihilated by all the annihilation operators, the vacuum transforming as the lowest weight state of the representation. We can also realize irreducible representations (IR) of $G_2$ in the fermionic Fock space. To do this, what is relevant is the identification of a state transforming as the lowest or highest weight state. From the form of the Cartan generators (10) and the property that the fermionic number operator $N_i = a_i^+ a_i$ has eigenvalues 0 or 1 on the Fock space, one finds that in the Fock space there are two states which can be candidate for the highest weight states for the trivial one-dimensional IR (in Dynkin notation $(0,0)$) and for the fundamental seven-dimensional IR (in Dynkin notation $(0,1)$). The highest weight state for the $(0,0)$ IR is

$$|0,0\rangle = |0\rangle + a_1^+ a_2^+ a_3^+ |0\rangle \tag{11a}$$

while the highest weight state of the $(0,1)$ IR is

$$|0,1\rangle = a_1^+ a_2^+ |0\rangle. \tag{11b}$$

By applying to the above state the generators corresponding to the simple negative roots (6*b*) in the sequence $F_2, F_1, F_2, F_2, F_1, F_2$ all the states of the representation are spanned. One can easily identify the states of the representation of $A_2 \subset G_2$ i.e.

$$7 = 3 + \bar{3} + 1$$

the IR **3** being spanned by $a_i^+|0\rangle$, the IR $\bar{\mathbf{3}}$ by $a_i^+ a_j^+ |0\rangle$ $(i \neq j)$ and the trivial representation by the state given in (11*a*).

Let us now define

$$d_i^+ = a_i^+ + \tfrac{1}{2}\varepsilon_{ijk} a_j a_k \qquad \text{and} \qquad d_i = a_i - \tfrac{1}{2}\varepsilon_{ijk} a_j^+ a_k^+ \tag{12}$$

which satisfy the following relations (and the HC relations)†

$$[d_i^+, d_i] = K_i \qquad [K_i, K_j] = 0 \tag{13a}$$

$$[K_i, d_j^+] = (3\delta_{ij} - 1) d_j^+ \qquad [K_i, d_j] = -(3\delta_{ij} - 1) d_j \tag{13b}$$

$$[[d_i^+, d_j], d_k^+] = 3\delta_{jk} d_i^+ \quad (i \neq j) \qquad [[d_i^+, d_j], d_k] = -3\delta_{ik} d_j \quad (i \neq j) \tag{13c}$$

$$(d_i^+)^3 = (d_i^+)^2 d_j = d_i^+ d_j d_i^+ = d_i^+ d_j^+ d_j d_i^+ = d_j^+ d_j^+ d_i^+ d_i^+ = d_i^+ d_j d_j^+ d_j d_i^+ = 0 \quad (i \neq j) \tag{13d}$$

$$\begin{aligned} d_i^+ d_j^+ d_k d_i^+ &= d_k^+ d_j d_i d_i = d_i^+ d_j d_k d_j^+ d_i^+ = d_i^+ d_j d_j^+ d_k d_i^+ = d_i^+ d_j d_j^+ d_k d_j d_i^+ \\ &= d_i^+ d_j d_k d_j^+ d_j d_i^+ = 0 \quad (i \neq j \neq k) \end{aligned} \tag{13e}$$

with $K_1 + K_2 + K_3 = 0$.

Now, forgetting the defining equations (12), we define objects satisfying (13) and we shall call them 'skedofermions' (or quasi-parafermions). Of course, one can generalize this definition to build a set of $n$ skedofermions in connection with any $A_n$ algebra.

In terms of skedofermions, we can obtain a realization of $G_2$ as

$$E_1 = \tfrac{1}{3}[d_1^+, d_2] \qquad \text{and} \qquad E_2 = d_2^+ \tag{14a}$$

$$F_1 = \tfrac{1}{3}[d_2^+, d_1] \qquad \text{and} \qquad F_2 = d_2 \tag{14b}$$

$$H_1 = \tfrac{1}{3}(K_1 - K_2) \qquad \text{and} \qquad H_2 = K_2. \tag{14c}$$

Now we can restate the previously obtained results on the IRs in terms of skedofermion space. In reference [5], it has been shown that a vacuum state can be introduced for parafermions. Then the Fock space is spanned by monomials in the creation parafer-

† One can find other relations of type (13*e, f, g*) using the defining equation (12). Actually, it is easy to convince oneself that monomials in $d_i^+$ and $d_i$ $(i = 1, 2, 3)$ are identically vanishing if their degree is sufficiently large. These relations have the property to be preserved by the $q$-deformation defined by equation (15).

mionic operators and the ring of parafermions is proved to be irreducible. A key point in these proofs is $(4.1)_1$ of [5], which, up to a numerical factor $\frac{2}{3}$, corresponds to our (13*c*). So, we can prove, along the same lines, analogous results for skedofermions. However, due to the peculiar realization of the generators of $G_2$ as commutator of skedofermions, the vacuum state cannot be a candidate to a lowest or highest weight state, which, as we have already remarked, is the really relevant condition to verify. One can check that if $|V\rangle$ is a vector in the Fock space of skedofermions, eigenvector of $K_1$ and $K_2$, such that $|\Omega\rangle = d_3^2|V\rangle \neq 0$, then due to (12) $|\Omega\rangle$ is the highest weight state of the representation (0, 1), up to an irrelevant constant shift in the Cartan generators eignevalues and all the states of the IR are obtained by applying the $F_i$ in the above sequence.

Now we define $q$-skedofermions $\psi_i^+$, $\psi_i$ (i.e. quantum analogue of skedofermions) by the following relations (and the HC relations) which have the property to reduce to (13) for $q=1$ and which are analogous of the equations defining $q$-fermions [2]:

$$[\psi_i^+, \psi_i] = [K_i]_{q^2} \tag{15a}$$

$$[K_i, \psi_j^+] = (3\delta_{ij} - 1)\psi_j^+ \qquad [K_i, \psi_j] = -(3\delta_{ij} - 1)\psi_j \tag{15b}$$

$$[K_i, K_j] = 0 \tag{15c}$$

$$[[\psi_i^+, \psi_j]_{q^{-2}}, \psi_k^+] = 3\delta_{jk} q^{2K_j}\psi_i^+ \qquad [[\psi_i^+, \psi_j]_{q^{-2}}, \psi_k] = -3\delta_{ik}\psi_j q^{2K_i} \qquad (i \neq j) \tag{15d}$$

$$(\psi_i^+)^3 = (\psi_i^+)^2\psi_j = \psi_i^+\psi_j\psi_i^+ = \psi_i^+\psi_j^+\psi_j\psi_i^+ = \psi_j^+\psi_j^+\psi_i^+\psi_i^+$$
$$= \psi_i^+\psi_j\psi_j^+\psi_j\psi_i^+ = 0 \qquad (i \neq j) \tag{15e}$$

$$\psi_i^+\psi_j^+\psi_k\psi_i^+ = \psi_k^+\psi_j\psi_i\psi_i = \psi_i^+\psi_j\psi_k\psi_j^+\psi_i^+ = \psi_i^+\psi_j\psi_j^+\psi_k\psi_i^+$$
$$= \psi_i^+\psi_j\psi_j^+\psi_k\psi_j\psi_i^+ = \psi_i^+\psi_j\psi_k\psi_j^+\psi_j\psi_i^+ = 0 \qquad (i \neq j \neq k) \tag{15f}$$

where

$$[K]_{q^2} = \frac{q^{2K} - q^{-2K}}{q^2 - q^{-2}} \tag{16}$$

and the $q$-commutator of two operators $A$ and $B$ is defined by

$$[A, B]_q = AB - qBA. \tag{17}$$

From (15*d*) and the identity

$$[A, B] = \frac{q^{-1}[A, B]_{q^2} + q[A, B]_{q^{-2}}}{q + q^{-1}} \tag{18}$$

it follows that

$$[[\psi_i^+, \psi_j], \psi_k^+] = 3\delta_{jk}\{K_j\}_{q^2}\psi_i^+ \qquad (i \neq j) \tag{19a}$$

$$[[\psi_i^+, \psi_j], \psi_k] = -3\delta_{ik}\psi_j\{K_i\}_{q^2} \qquad (i \neq j) \tag{19b}$$

where

$$\{K\}_{q^2} = \frac{q^{2K+1} + q^{-2K-1}}{q + q^{-1}}. \tag{20}$$

Notice that (19*a*, *b*), up to a numerical factor $\frac{2}{3}$, are also satisfied by $q$-fermions replacing the operator $K_i$ by the fermionic number operator $N_i$.

It is well known that ordinary fermions satisfy the relations of the $q$-Clifford algebra due to the property that the square of $N_i$ is equal to $N_i$. As $K_i^2 \neq K_i$, $q$-skedofermions are different from skedofermions. It is even almost evident that replacing in (12) the fermions by $q$-fermions does not define $q$-skedofermions; one can wonder if skedofermions can be realized in terms of more complicated expressions of $q$-fermions. This is not apparently the case (even we do not have a general proof) if we require in particular (15) to hold.

Let us recall the definition of a quantum enveloping algebra $U_q(G)$ associated with a simple Lie algebra $G$ of rank $r$. Mathematically, the quantum enveloping algebra $U_q(G)$ is a Hopf algebra with unit 1 and generators $E_i$, $F_i$, $H_i$ ($1 \leqslant i \leqslant r$) defined through the commutation relations in the Chevalley basis

$$[H_i, H_j] = 0 \qquad [H_i, E_j] = a_{ij}E_j \qquad [H_i, F_j] = -a_{ij}F_j \tag{21$a, b, c$}$$

$$[E_i, F_j] = \delta_{ij}\frac{q^{2d_iH_i} - q^{-2d_iH_i}}{q^{2d_i} - q^{-2d_i}} \tag{21$d$}$$

and the quantum Serre–Chevalley relations (for $i \neq j$)

$$\sum_{0 \leqslant n \leqslant 1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q^{2d_i}} (E_i)^{1-a_{ij}-n} E_j (E_i)^n = 0 \tag{22$a$}$$

$$\sum_{0 \leqslant n \leqslant 1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q^{2d_i}} (F_i)^{1-a_{ij}-n} F_j (F_i)^n = 0 \tag{22$b$}$$

where $(a_{ij})$ ($1 \leqslant i, j \leqslant r$) is the Cartan matrix of the Lie algebra $G$, and $d_i$ are non-zero integers, with greatest common divisor equal to one, such that $d_ia_{ij} = d_ja_{ji}$. Note that in the case of $G_2$ with Cartan matrix (9), one has $d_1 = 3$ and $d_2 = 1$.

The $q$-binomial coefficients $[^m_n]_q$ are defined by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q![n]_q!} \qquad \text{with } [m]_q! = [m]_q \ldots [1]_q \text{ and } [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{23}$$

One needs also to introduce a comultiplication $\Delta$, a co-unit $\varepsilon$ and an antipode $S$ such that

$$\begin{aligned} &\Delta(H_i) = 1 \otimes H_i + H_i \otimes 1 \\ &\Delta(E_i) = E_i \otimes q^{-H_i} + q^{H_i} \otimes E_i \qquad \Delta(F_i) = F_i \otimes q^{-H_i} + q^{H_i} \otimes F_i \\ &\varepsilon(E_i) = \varepsilon(F_i) = \varepsilon(H_i) = 0 \qquad \varepsilon(1) = 1 \\ &S(E_i) = -q^{-H_i}E_iq^{H_i} \qquad S(E_i) = -q^{-H_i}E_iq^{H_i} \qquad S(H_i) = -H_i. \end{aligned} \tag{24}$$

Then we can prove the following statement:

*Proposition.* A realization of the generators of $U_q(G_2)$ is given by

$$E_1 = \frac{1}{\sqrt{3}\sqrt{q^4 + q^{-4} + 1}} [\psi_1^+, \psi_2]_{q^{-2}} \qquad E_2 = \psi_2^+ \tag{25$a$}$$

$$F_1 = \frac{1}{\sqrt{3}\sqrt{q^4 + q^{-4} + 1}} [\psi_2^+, \psi_1]_{q^2} \qquad F_2 = \psi_2 \tag{25$b$}$$

$$H_1 = \tfrac{1}{3}(K_1 - K_2) \qquad H_2 = K_2. \tag{25$c$}$$

*Proof.* Equations (21$a$–$c$) are just the defining equations (15$b$, $c$).

Equation (21$d$) for $i = j = 2$ reduces to (15$a$) with $i = 2$ and finally (21$d$) for $i = j = 1$ is a consequence of equations (15$a$–$d$) and of the identity $q^6 - q^{-6} = (q^2 - q^{-2})(q^4 + q^{-4} + 1)$.

Now we have to verify the quantum Serre–Chevalley relations (22). Equation (22*a*) reads for $a_{12}=-1$

$$\sum_{n=0}^{2}(-1)^n\begin{bmatrix}2\\n\end{bmatrix}_{q^6}(E_1)^{2-n}E_2(E_1)^n=[E_1,[E_1,E_2]_{q^{-6}}]_{q^6}$$
$$=E_1^2E_2-(q^6+q^{-6})E_1E_2E_1+E_2E_1^2 \qquad (26)$$

and for $a_{21}=-3$

$$\sum_{n=0}^{4}(-1)^n\begin{bmatrix}4\\n\end{bmatrix}_{q^2}(E_2)^{4-n}E_1(E_2)^n=[E_2,[E_2,[E_2,[E_2,E_1]_{q^{-2}}]_{q^2}]_{q^{-2}}]_{q^2}$$
$$=E_2^4E_1-\begin{bmatrix}4\\1\end{bmatrix}_{q^2}E_2^3E_1E_2+\begin{bmatrix}4\\2\end{bmatrix}_{q^2}E_2^2E_1E_2^2-\begin{bmatrix}4\\3\end{bmatrix}_{q^2}E_2E_1E_2^3+E_1E_2^4. \qquad (27)$$

Now, if we write the RHS of equations (26) and (27) in terms of $\psi_1^+$, $\psi_2^+$ and $\psi_2$, one can easily prove that each term identically vanishes using equations (15*e*) and (15*f*), so that the quantum Serre–Chevalley relations are satisfied.

The proof of (22*b*) is evidently in the same spirit, using the HC relations of (15*e*) and (15*f*). In the Fock space of $q$-skedofermions one can realize the fundamental representation of $U_q(G_2)$ by introducing the $q$-analogue of the $|V\rangle$ vector such that $\psi_3^2|V\rangle\neq 0$. From (15) one can see that this vector is annihilated by $E_1$, $E_2$ and that non-vanishing vectors can only be obtained by applying the $F_i$ $(i=1,2)$ generators in the above specified sequence, so obtaining a seven-dimensional representation.

Notice that, in this construction, a key point is the definition of the $q$-skedofermions given by (15). Of course, this definition is not unique as the requirement to recover equations (13) for $q=1$ is indeed very weak and the criterion of 'close analogy' of the deformation of fermions is mathematically rather badly defined. We have found a consistent deformation which is suitable for the construction of $U_q(G_2)$. Moreover, one can use these $q$-skedofermions to build a new realization of $U_{q'}(A_2)$ (with $q'=q^3$) which has the following structure:

$$E_1=\frac{1}{\sqrt{3}\sqrt{q^4+q^{-4}+1}}[\psi_1^+,\psi_2]_{q^{-2}} \qquad E_2=\frac{1}{\sqrt{3}\sqrt{q^4+q^{-4}+1}}[\psi_2^+,\psi_3]_{q^{-2}} \qquad (28a)$$

$$F_1=\frac{1}{\sqrt{3}\sqrt{q^4+q^{-4}+1}}[\psi_2^+,\psi_1]_{q^{2}} \qquad F_2=\frac{1}{\sqrt{3}\sqrt{q^4+q^{-4}+1}}[\psi_3^+,\psi_2]_{q^{2}} \qquad (28b)$$

$$H_1=\tfrac{1}{3}(K_1-K_2) \qquad H_2=\tfrac{1}{3}(K_2-K_3). \qquad (28c)$$

Using (15), it is easy to verify that (21) hold and that the quantum Serre–Chevalley relations (22) are satisfied.

## References


[1] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
Chang-Pu Sun and Hong-Chen Fu 1989 *J. Phys. A: Math. Gen.* **22** L983

[2] Hayashi T 1990 *Commun. Math. Phys.* **127** 129

[3] Frappat L, Sciarrino A and Sorba P 1991 *J. Phys. A: Math. Gen.* **24** L179

[4] Reshetekhin N Yu 1988 Quantified universal enveloping algebras, the Yang–Baxter equation and invariants of links, II *LOMI preprint*

[5] Ohnuki Y and Kamefuchi S 1982 *Quantum Field Theory and Parastatistics* (Berlin: Springer)

[6] Floreanini R and Vinet L 1990 *J. Phys. A: Math. Gen.* **23** L1019

[7] Sciarrino A 1989 *J. Math. Phys.* **30** 1674